Divergence and Curl of a Vector Function

- This unit is based on Section 9.7 , Chapter 9.
- All assigned readings and exercises are from the textbook
- Objectives:

  Make certain that you can define, and use in context, the terms, concepts and formulas listed below:

  1. find the divergence and curl of a vector field.
  2. understand the physical interpretations of the Divergence and Curl.
  3. solve practical problems using the curl and divergence.
- Reading: Read Section 9.7, pages 483-487.
- Exercises: Complete problems

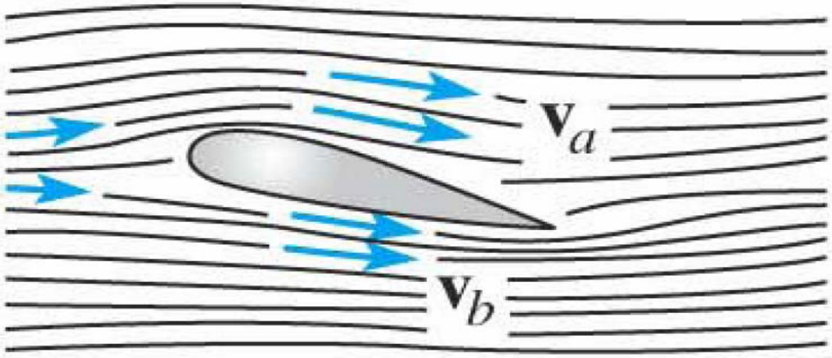
- Prerequisites: Before starting this Section you should . . .
  - ✓ be familiar with the concept of partial differentiation
  - ✓ be familiar with vector dot and cross multiplications
  - ✓ be familiar with 3D coordinate system

Differentiation of vector fields

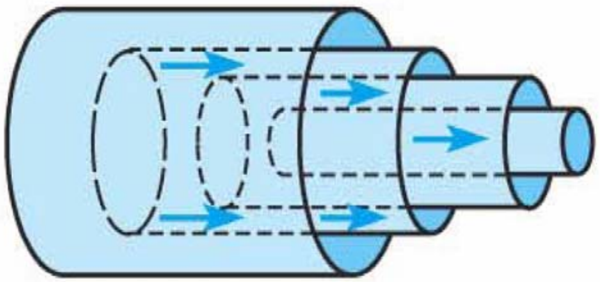
- **Example of a vector field**: Suppose fluid moves down a pipe, a river flows, or the air circulates in a certain pattern. The velocity can be different at different points and may be at different time.
- The velocity vector **F** gives the direction of flow and speed of flow at every point.
- **Applications of Vector Fields**:
  - Mechanics
  - Electric and Magnetic fields
  - Fluids motions
  - Heat transfer
- There are two kinds of differentiation of a vector field **F**(x,y,z):

1. divergence (div $\mathbf{F} = \nabla \cdot \mathbf{F}$) and
2. curl (curl $\mathbf{F} = \nabla \times \mathbf{F}$)

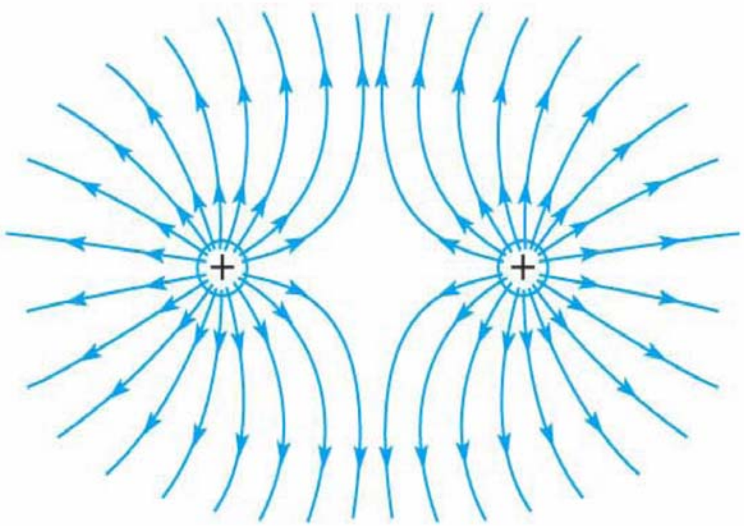
Examples of Vector Fields



(a) Airflow around an airplane wing



(b) Laminar flow of blood in an artery; cylindrical layers of blood flow faster near the center of the artery



(d) Lines of force around two equal positive charges

The Divergence of a Vector Field

- Consider the vector fields

Vector function with two variable:

$$\vec{\mathbf{F}}(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}}$$

Vector function with three variable:

$$\vec{\mathbf{F}}(x, y, z) = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}}$$

- We define the divergence of **F**

$$Div\ \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- In terms of the differential operator ∇, the divergence of **F**

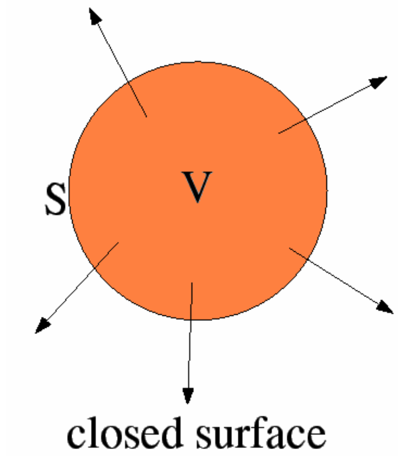
$$\boxed{Div\ \vec{F} = \nabla \bullet \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

- **A key point**: **F** is a vector and the divergence of **F** is a scalar.

Example: $\vec{F} = 4xy\hat{i} + (2x^2 + 2yz)\hat{j} + 3(z^2 + y^2)\hat{k}$, **Find** $\nabla . \vec{F}$

Divergence

- Divergence is the outflow of flux from a small closed surface area (per unit volume) as volume shrinks to zero.



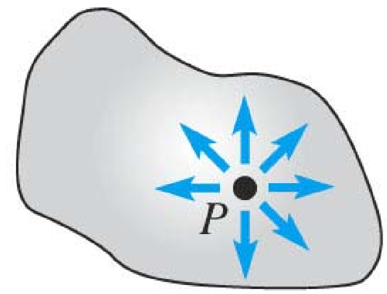
- Air leaving a punctured tire: Divergence is positive, as closed surface (tire) exhibits net outflow

- The divergence measures sources and drains of flow:

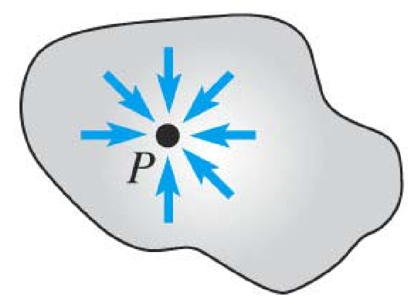
$$\nabla \cdot \mathbf{F}(@P) > 0 \Rightarrow \textbf{source}$$

$$\nabla \cdot \mathbf{F}(@P) < 0 \Rightarrow \textbf{sink}$$

$$\nabla \cdot \mathbf{F}(@P) = 0 \Rightarrow \textbf{no source or sink}$$



(a) div $\mathbf{F}(P) > 0$; P a source

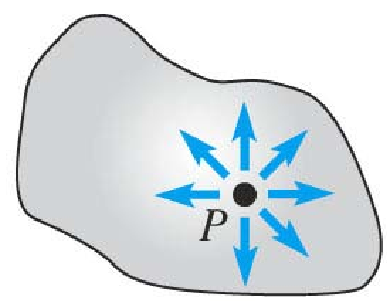


(b) div $\mathbf{F}(P) < 0$; P a sink

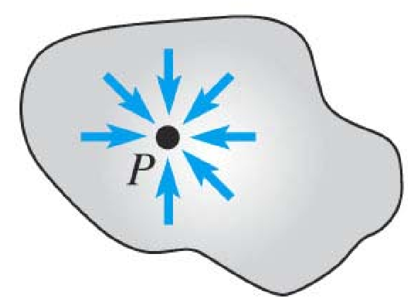
Physical Interpretation of the Divergence

- Consider a vector field **F** that represents a fluid velocity:

  The divergence of **F** at a point in a fluid is a measure of the rate at which the fluid is flowing away from or towards that point.
- A positive divergence is indicating a flow away from the point.
- Physically divergence means that either the fluid is expanding or that fluid is being supplied by a source external to the field.
- The lines of flow diverge from a source and converge to a sink.
- If there is no gain or loss of fluid anywhere then div **F** = 0. Such a vector field is said to be solenoidal.
- The divergence also enters electrical engineering topics such as electric and magnetic fields:
  - For a magnetic field: $\nabla \cdot \mathbf{B} = 0$, that is there are no sources or sinks of magnetic field, a solenoidal filed.
  - For an electric field: $\nabla \cdot \mathbf{E} = \rho/\varepsilon$, that is there are sources of electric field..



(a) div $\mathbf{F}(P) > 0$; P a source



(b) div $\mathbf{F}(P) < 0$; P a sink

The Curl of a Vector Field

- Consider the vector fields

$$\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}}$$

- The curl of **F** is another vector field defined as:

$$\boxed{\textbf{curl } \vec{\mathbf{F}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}}$$

- In terms of the differential operator ∇, the curl of **F**

$$\boxed{\textbf{Curl } \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}}} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$$

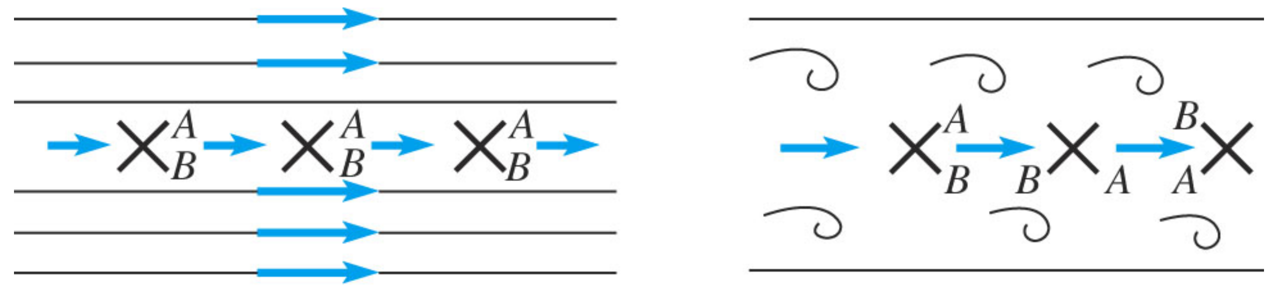
- **A key point**: **F** is a vector and the **curl** of **F** is a **vector**.

Example: $\vec{F} = 4xy\hat{i} + (2x^2 + 2yz)\hat{j} + 3(z^2 + y^2)\hat{k}$, **Find** $\nabla \times \vec{F}$

Physical Interpretation of the Curl

- Consider a vector field **F** that represents a fluid velocity:
  The curl of **F** at a point in a fluid is a measure of the rotation of the fluid.
- If there is no rotation of fluid anywhere then $\nabla \times \mathbf{F} = 0$. Such a vector field is said to be irrotational or **conservative**.
- For a 2D flow with **F** represents the fluid velocity, $\nabla \times \mathbf{F}$ is perpendicular to the motion and represents the direction of axis of rotation.

Related Course:
ENGR361



(a) Irrotational flow (b) Rotational flow

- The curl also enters electrical engineering topics such as electric and magnetic fields:
- A magnetic field (denoted by **H**) has the property $\nabla \times \mathbf{H} = \mathbf{J}$.
- An electrostatic field (denoted by **E**) has the property $\nabla \times \mathbf{E} = 0$, an irrotational (conservative) field. **Related Course**: Elec 251/351

Further properties of the vector differential operator ∇

1) $\mathbf{div}\left[\mathbf{grad}\ f(x,y,z)\right] = \nabla \bullet \nabla f = \nabla^2 f;$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

∇^2 is called the Laplacian operator

2) $\nabla\left[f(r)g(r)\right] = g\nabla f + f\nabla g$

3) $\nabla \bullet \left[f(r)\vec{F}(r)\right] = f\nabla \bullet \vec{F} + \vec{F} \bullet \nabla f$

4) $\nabla \times \left[f(r)\vec{F}(r)\right] = f\nabla \times \vec{F} + (\nabla f) \times \vec{F}$

5) $\nabla \bullet \left[\vec{F}(r) \times \vec{G}(r)\right] = \vec{G} \bullet (\nabla \times \vec{F}) - \vec{F} \bullet (\nabla \times \vec{G})$

6) $\mathbf{div[curl}\ \vec{F}(r)\mathbf{]} = \nabla \bullet (\nabla \times \vec{F}) = 0$

7) $\mathbf{curl[grad}\ f(r)\mathbf{]} = \nabla \times (\nabla f) = 0$

Verification Examples: $f = x^2y^2z^3$; $\vec{F} = \langle x^2y, xy^2z, -yz^2 \rangle$

Vector Calculus and Heat Transfer

- Consider a solid material with **density** ρ, **heat capacity** *c* , the **temperature distribution** T(x,y,z,t) and **heat flux vector q.**
- **conservation of heat energy**

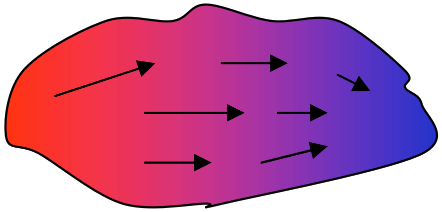
$$\frac{\partial}{\partial t}(\rho c T) + \nabla \cdot \mathbf{q} = 0$$

- In many cases the heat flux is given by Fick's law

$$\mathbf{q} = -k \nabla T$$

- Which results in heat equation:

$$\boxed{\frac{\partial T}{\partial t} = \kappa \nabla^2 T,}$$



- Related Course: MECH352

Vector Calculus and Fluid Mechanics

- **Conservation of Mass**:

  Let

  ρ be the **fluid density** and

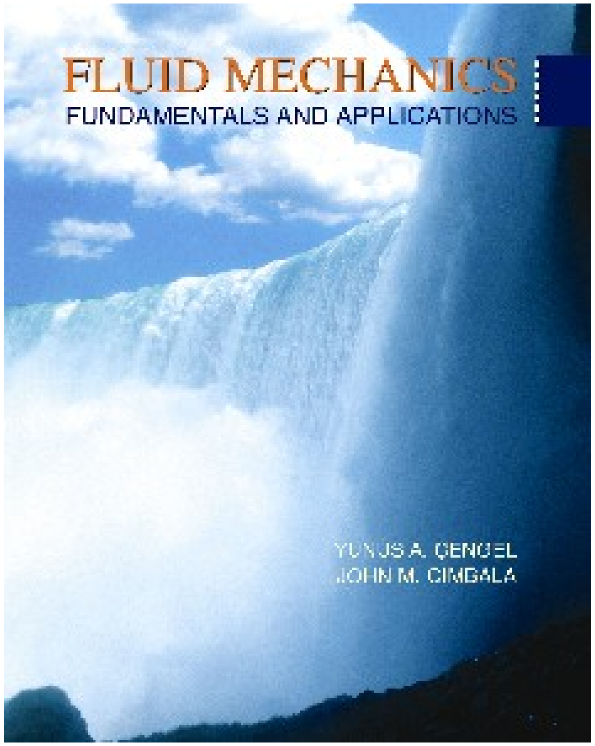
  v be the **fluid velocity.**

  Conservation of mass in a volume gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Which can be written as

$$\boxed{\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0}$$



- **Related Course**: ENGR361

Vector Calculus and Electromagnetics

- **Maxwell equations in free space**
  - Maxwell Equations describe the transmission of information (internet data, TV/radio program, phone,…) using wireless communication.

$$\nabla \cdot \mathbf{E} = \rho_v / \varepsilon_0, \quad \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \partial \mathbf{E} / \partial t$$

- Solutions of this equations are essential for the analysis, design and advancement of wireless devices and system, high-speed electronics, microwave imaging, remote sensing, …etc.
- **Related Courses**: ELEC251, ELEC351, ELEC353, ELEC453, ELEC 456, ELEC 457

Magneto-static Field Example

Magneto-static Field is an example of rotational field

$$\nabla \times \mathbf{B} = \mathbf{J}$$

$$\nabla \times \mathbf{B} = 0, \text{ outside the cable}$$

$$\nabla \times \mathbf{B} \neq 0, \text{ inside the cable}$$